# LECTURE 17

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Last time we learned the formula to compute the area of a parametrized surface $\mathbf{r} : \Omega \to \mathbb{R}^3$, which is

$$\iint_\Omega \|\mathbf{r}_u \times \mathbf{r}_v\| \, du \, dv. \tag{1}$$

Note that it is very similar to how we compute the length of a parametrized curve $\mathbf{r} : [a,b] \to \mathbb{R}^n$

$$\int_C ds = \int_a^b \|\mathbf{r}'(t)\| dt,$$

where $s$ is the parameter for the unit speed parametrization. Motivated by this, in Equation (1) let us also give $\|\mathbf{r}_u \times \mathbf{r}_v\| \, du \, dv$ a name and call it $d\sigma$.

Once you learn about how to find the area of a surface, you know how to integrate a function on the surface, because finding the area is nothing but integrating the constant function 1.

**Example 1.** We compute the integral of $G(x,y,z) = x^2$ over the cone surface $S$ defined by:

$$z = \sqrt{x^2 + y^2}, \quad 0 \le z \le 1.$$

First, we parametrize the cone by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = r, \quad \text{where } 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

The surface element $d\sigma$ is:

$$d\sigma = \sqrt{2} r \, dr \, d\theta.$$

The integral becomes:

$$\iint_S x^2 \, d\sigma = \int_0^{2\pi} \int_0^1 r^2 \cos^2\theta \cdot \sqrt{2} r \, dr \, d\theta = \frac{\pi\sqrt{2}}{4}.$$

Another thing we can do about curves is to compute the circulation and flux of a vector field defined in an open neighborhood of the curve. While the former does not make sense for a surface (as surfaces have tangent planes instead of tangent vectors), the latter does.

A **normal vector field** on a surface $S \subset \mathbb{R}^3$ is a continuous assignment of a unit vector $\mathbf{n}(p)$ orthogonal to the tangent plane at each point $p \in S$. The surface $S$ is **orientable** if such a normal vector field exists globally, meaning there is a consistent choice of "up" or "down" direction across the entire surface. The choice of $\mathbf{n}$ determines an **orientation** of $S$; for example, the standard orientation of the sphere is given by the outward-pointing normal. If no such continuous $\mathbf{n}$ exists, the surface is **non-orientable**. Formally, an orientable surface admits two possible orientations, corresponding to $\mathbf{n}$ and $-\mathbf{n}$. A prototypical example of a non-orientable surface is the Möbius strip.

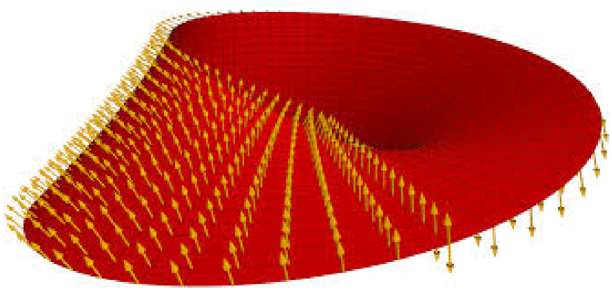

FIGURE 1. Möbius strip

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A parametrized (smooth) surface $\mathbf{r} : \Omega \to S \subset \mathbb{R}^3$, where $\Omega \subset \mathbb{R}^2$, naturally carries an orientation. The cross product $\mathbf{r}_u \times \mathbf{r}_v$ yields a normal vector to the tangent plane at each point, and if $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$, we may define the **unit normal vector field**

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}.$$

This choice of $\mathbf{n}$ induces a consistent orientation on $S$ as long as $\mathbf{r}_u \times \mathbf{r}_v$ varies continuously. Reversing the parametrization (e.g., swapping $u$ and $v$) flips the normal vector, giving the opposite orientation. Thus, a regular parametrization $\mathbf{r}$ automatically provides an orientation for $S$.

Now, given a vector field $\mathbf{F}$ defined over an open neighborhood of an oriented surface $S$. We define the **flux** of $\mathbf{F}$ through $S$ by

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma.$$

In the case of a parametrized surface $\mathbf{r} : \Omega \to \mathbb{R}^3$, the above simplies to

$$\iint_\Omega \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) du dv.$$

**Example 2.** Find the flux of $\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}$ through the parabolic cylinder $y = x^2$, $0 \leq x \leq 1$, $0 \leq z \leq 4$. The surface is oriented such that the normal vector has a positive $x$-component.

The parabolic cylinder can be parameterized using $x$ and $z$ as parameters:

$$\mathbf{r}(x,z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}, \quad 0 \leq x \leq 1, \quad 0 \leq z \leq 4.$$

Compute the tangent vectors:

$$\mathbf{r}_x = \frac{\partial \mathbf{r}}{\partial x} = \mathbf{i} + 2x\mathbf{j}, \quad \mathbf{r}_z = \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k}.$$

The normal vector is given by the cross product $\mathbf{r}_x \times \mathbf{r}_z$:

$$\mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\mathbf{i} - \mathbf{j}.$$

The given orientation requires the normal vector to have a positive $x$-component. Since the $x$-component here is $2x$ (which is positive for $x \in [0,1]$), we use this normal vector as is.

The flux of $\mathbf{F}$ through the surface $S$ is given by:

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_\Omega \mathbf{F}(\mathbf{r}(x,z)) \cdot (\mathbf{r}_x \times \mathbf{r}_z)\, dx\, dz,$$

where $\Omega = [0,1] \times [0,4]$. Therefore, the answer is

$$\iint_\Omega (2x^3 z - x)\, dx\, dz = \int_{x=0}^{1} \int_{z=0}^{4} (2x^3 z - x)\, dz\, dx = 2.$$